A Summary of Basic Vibration Theory

There are many different approaches to the analysis of vibration and it rarely happens that two individuals will be familiar with the same notation never mind the same perspective. This situation has arisen mainly because there is no universal approach to vibration problems rather different problem areas are more amenable to solution using particular techniques. These notes are concerned with a spring-mass system with viscous damping which is constrained to vibrate as a single degree-of-freedom system. There are links (in blue) to WWW pages, some with animations.

Spring-mass system with viscous damping

The basic elements of the system are a massless spring having a stiffness k, a linear massless viscous damper having a viscous damping coefficient c and a rigid mass m. The system is further considered to have one degree of freedom so that the value of the co-ordinate x(t) uniquely defines the position of the whole system. It is conventional and helpful to define x(t) with respect to the static equilibrium position of the system so that the same equations apply whether the system is horizontal or vertical. The next step is to consider the forces acting on each of the elements when the mass is at x(t) i.e., displaced from the equilibrium position, see Figure 2.

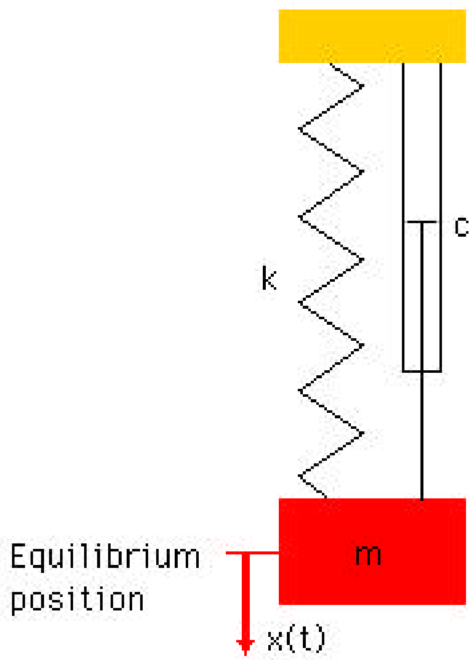


Figure1 Basic single degree of freedom system.

This allows the application of Newton's second law of motion to give,

$$m\ddot{x}(t) = -kx(t) - c\dot{x}(t) \quad .... \quad (1)$$

and hence

$$m\ddot{x}(t) + c\,\dot{x}(t) + kx(t) = 0 \quad .... \quad (2)$$

The signs in equation (1) are not fortuitous but are consistent with the definition of x(t) as shown in Figure 1. Thus x(t) is defined as positive downwards and therefore the mass times the acceleration in the positive direction, $\ddot{x}(t)$, is equal to the total force acting on the mass in the positive direction. In this case this force consists of two parts both negative.

When a solution is attempted of equation (1) then various methods are available. For the condition presented, i.e. no excitation force, then two initial conditions, $x(0)$ and $\dot{x}(t)$, need to be specified.

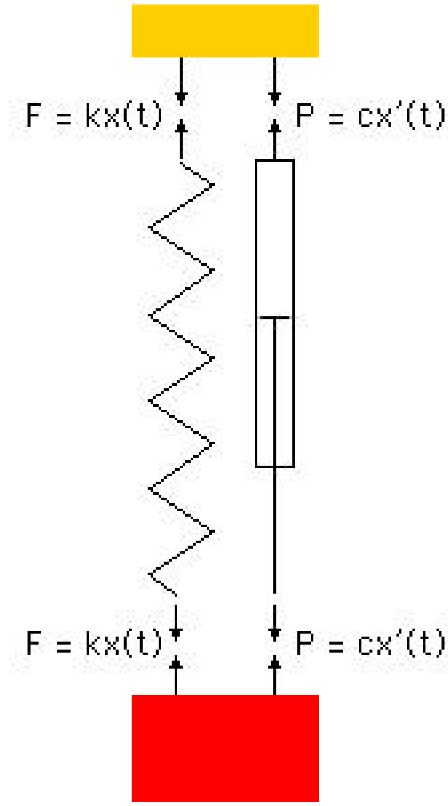


Figure 2 Forces acting on the mass.

<u>Laplace transform solution.</u>

Taking Laplace transforms of equation (1) yields,

$$m[-\dot{x}(0) - sx(0) + s^2X(s)] + c[-x(0) + sX(s)] + kX(s) = 0$$

rearranging this gives

$$X(s) = \frac{m(\dot{x}(0) + sx(0)) + cx(0)}{ms^2 + cs + k} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

<u>Case of no damping, i.e. c=0</u>

Consider first, the case when there is no damping, i.e. c = 0. Equation (3) then becomes,

$$X(s) = \frac{m(\dot{x}(0) + sx(0))}{ms^2 + k} = \frac{\dot{x}(0)}{s^2 + k/m} + \frac{sx(0)}{s^2 + k/m} \quad \ldots\ldots\ldots\ldots \quad (4)$$

Now taking inverse Laplace transforms gives

$$x(t) = \frac{\dot{x}(t)}{\sqrt{\frac{k}{m}}} \sin\left(\sqrt{\frac{k}{m}}\, t\right) + x(0)\cos\left(\sqrt{\frac{k}{m}}\, t\right) \quad \ldots\ldots\ldots\ldots\ldots \quad (5)$$

If the system is given an initial displacement and no initial velocity,
i.e. $x(0) = X_0$ and $\dot{x}(t) = 0$ then

$$x(t) = X_0 \cos\left(\sqrt{\frac{k}{m}}\, t\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

this is a non decaying sinusoidal vibration of frequency √k/m and is conventionally termed the undamped natural frequency and is denoted by ω_n so that

$$\omega_n = \sqrt{\frac{k}{m}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

similarly with an initial velocity alone or with both initial velocity and displacement a sinusoidal vibration of frequency ω_n would be obtained.

Equations in non-dimensional form

It is common to write the basic equation of motion in terms of ω_n and another parameter ξ, the viscous damping ratio, which is defined as,

$$\xi = \frac{c}{2\sqrt{km}}$$ (The significance of ξ will become apparent later).

Thus if equation (2) is divided throughout by m we obtain,

$$\ddot{x}(t) + 2\xi\omega_n \dot{x}(t) + \omega_n^2 x(t) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (8)$$

If Laplace transforms are taken we obtain

$$X(s) = \frac{\dot{x}(0) + 2\xi\omega_n x(0)}{s^2 + 2\xi\omega_n s + \omega_n^2} + \frac{s x(0)}{s^2 + 2\xi\omega_n s + \omega_n^2} \quad \ldots\ldots\ldots\ldots \quad (9)$$

Now the form of solution depends on the value of ξ so that different solutions are obtained depending on whether $\xi < 0$, $\xi = 0$ or $\xi > 0$. We have already considered the case where c and hence ξ is zero. The solution was a non-decaying oscillation of frequency ω_n.

Case when $\xi < 1$

When $\xi < 1$ the inverse Laplace transform of equation (9) gives,

$$x(t) = e^{-\xi\omega_n t}\left[x(0)\cos\omega_n\sqrt{(1-\xi^2)}t + \frac{[\dot{x}(0)+\xi\omega_n x(0)]\sin\omega_n\sqrt{(1-\xi^2)}t}{\omega_n\sqrt{(1-\xi^2)}}\right] \quad \text{........... (10)}$$

This is an exponentially decaying oscillation. If again, the specific example of an initial displacement and no initial velocity is taken,

i.e. $x(0) = X_0$ and $\dot{x}(t) = 0$ then

$$x(t) = X_0 e^{-\xi\omega_n t}\left[\cos\omega_n\sqrt{(1-\xi^2)}t + \frac{\xi\sin\omega_n\sqrt{(1-\xi^2)}t}{\sqrt{(1-\xi^2)}}\right] \quad \text{........ (11)}$$

A typical variation of x(t) with t is shown in Figure 3. The greater the value of ξ the more rapid the decay of the oscillation. As $\xi \to 1$ then the solution tends to have no oscillation. In fact $\xi = 1$ is the lowest value of ξ which does not give any oscillation and this is thus termed the critical damping ratio. From the definition of ξ ($= c/2\sqrt{km}$) it is clear that when $\xi = 1$ the critical value of $c_c = 2\sqrt{km}$, i.e. $\xi = c/c_c$.

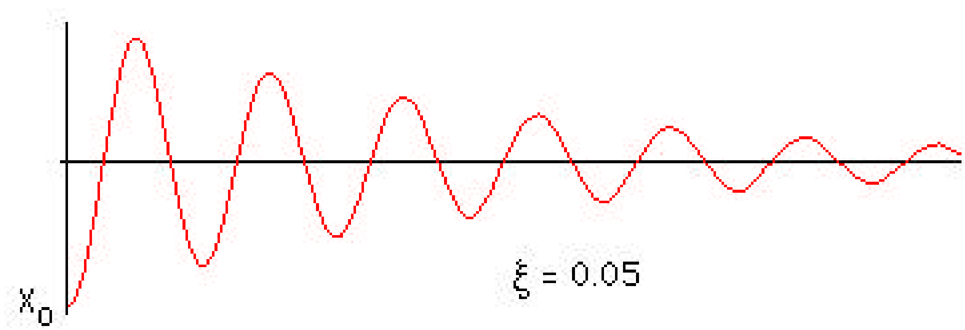


Figure 3 Typical vibration with decay.

<u>Case when $\xi = 1$</u>

When x = 1 the inverse Laplace transform of equation (9) gives,

$$x(t) = x(0)e^{-\omega_n t} + [\dot{x}(0) + \omega_n x(0)]te^{-\omega_n t} \quad \text{............. (12)}$$

This is a non oscillatory motion and for the case considered previously, i.e. $x(0) = X_0$ and $\dot{x}(t) = 0$ then

$$x(t) = x(0)\left(1 + \omega_n t\right)e^{-\omega_n t} \quad \text{.......................... (13)}$$

A typical variation of x(t) with t is shown in Figure 4. If the value of ξ exceeds 1.0 then another mathematical solution is obtained.

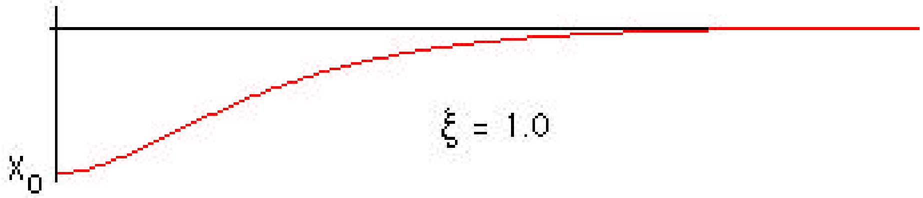


Figure 4 Critically damped decay.

Case when $\xi > 1$

When $\xi > 1$ the inverse Laplace transform of equation (9) gives,

$$x(t) = e^{-\xi\omega_n t}\left[x(0)\cosh\omega_n\sqrt{(\xi^2-1)}t + \frac{[\dot{x}(0)+\xi\omega_n x(0)]\sinh\omega_n\sqrt{(\xi^2-1)}t}{\omega_n\sqrt{(\xi^2-1)}}\right] \quad \text{......... (14)}$$

This is a non-oscillatory decaying motion and for the case considered previously, i.e. $x(0) = X_0$ and $\dot{x}(t) = 0$ then

$$x(t) = X_0 e^{-\xi\omega_n t}\left[\cosh\omega_n\sqrt{(\xi^2-1)}t + \frac{\xi\sinh\omega_n\sqrt{(\xi^2-1)}t}{\sqrt{(\xi^2-1)}}\right] \quad \text{.... (15)}$$

Typical variations of x(t) with t for various values of ξ are shown in Figure 5.

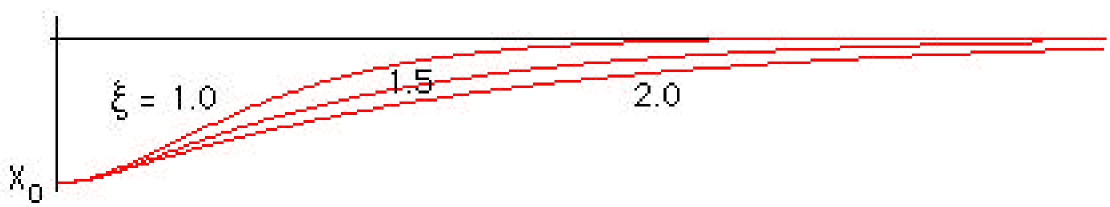


Figure 5 Decays for various damping ratios.

For practical engineering structures it would be extremely unusual for ξ to exceed unity and thus any transient vibration is normally oscillatory. It is possible from measured transients to calculate ξ. This approach uses the logarithmic decrement.

Logarithmic decrement.

For $\xi<1$ and free motion, i.e. no exciting force, the general solution is of the form

$$x(t) = e^{-\xi\omega_n t}\left[A\cos\omega_n\sqrt{(1-\xi^2)}\,t + B\sin\omega_n\sqrt{(1-\xi^2)}\,t\right] \quad \text{........ (16)}$$

where $A = X_0$ and $B = \dfrac{\dot{x}(0) + \xi\omega_n X_0}{\omega_n\sqrt{(1-\xi^2)}}$.

Thus for an arbitrary initial conditions a solution of the form given in equation (16) will apply. Now equation (16) may be manipulated as follows.

$$x(t) = e^{-\xi\omega_n t}\sqrt{(A^2 - B^2)}\left[\frac{A}{\sqrt{(A^2 - B^2)}}\cos\omega_n\sqrt{(1-\xi^2)}\,t + \frac{B}{\sqrt{(A^2 - B^2)}}\sin\omega_n\sqrt{(1-\xi^2)}\,t\right]$$

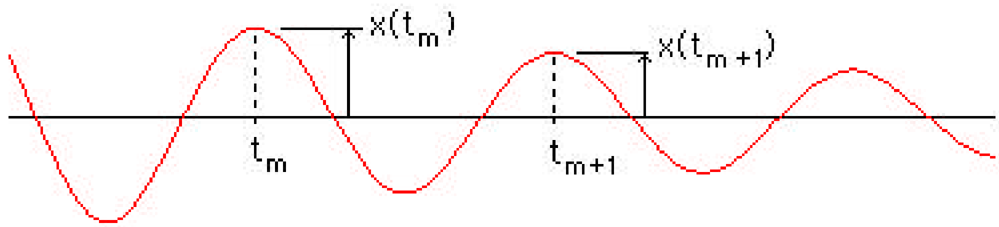
$$= e^{-\xi\omega_n t}\sqrt{(A^2 - B^2)}\left[\sin\varphi\cos\omega_n\sqrt{(1-\xi^2)}\,t + \cos\varphi\sin\omega_n\sqrt{(1-\xi^2)}\,t\right]$$

$$= Ce^{-\xi\omega_n t}\sin\left[\omega_n\sqrt{(1-\xi^2)}\,t + \varphi\right] \quad \text{...................... (17)}$$

where $\tan\varphi = A/B$ and $C = \sqrt{(A^2 + B^2)}$

Equation (17) represents a decaying oscillation of frequency $\omega_n\sqrt{(1-\xi^2)}$. Thus the damped natural frequency $\omega_D = \omega_n\sqrt{(1-\xi^2)}$.

Now consider successive maximum amplitudes



$\dfrac{dx(t)}{dt} = 0$ at a maximum. Differentiating equation (17)

$$\frac{dx(t)}{dt} = C\left[-\xi e^{-\xi\omega_n t}\sin\left[\omega_n\sqrt{(1-\xi^2)}\,t + \varphi\right] + \omega_n\sqrt{(1-\xi^2)}\,e^{-\xi\omega_n t}\cos\left[\omega_n\sqrt{(1-\xi^2)}\,t + \varphi\right]\right]$$

this is zero when

$$-\xi\sin\left[\omega_n\sqrt{(1-\xi^2)}\,t + \varphi\right] + \omega_n\sqrt{(1-\xi^2)}\cos\left[\omega_n\sqrt{(1-\xi^2)}\,t + \varphi\right] = 0$$

i.e. when $\tan\left[\omega_n\sqrt{(1-\xi^2)}t+\varphi\right]=\dfrac{\sqrt{(1-\xi^2)}}{\xi}$

which is when

$$\omega_n\sqrt{(1-\xi^2)}t+\varphi=\tan^{-1}\left[\frac{\sqrt{(1-\xi^2)}}{\xi}\right]+n\pi$$

This represents alternate max/min. Consider any two successive maxima or minima.

Then $\omega_n\sqrt{(1-\xi^2)}t_m+\varphi=\tan^{-1}\left[\dfrac{\sqrt{(1-\xi^2)}}{\xi}\right]+2m\pi$ .... (a)

and $\omega_n\sqrt{(1-\xi^2)}t_{m+1}+\varphi=\tan^{-1}\left[\dfrac{\sqrt{(1-\xi^2)}}{\xi}\right]+2(m+1)\pi$ .... (b)

(b) - (a) gives $t_{m+1}-t_m=\dfrac{2\pi}{\omega_n\sqrt{(1-\xi^2)}}$ .... (c)

Now $x(t_m)=Ce^{-\xi\omega_n t_m}\sin\left[\omega_n\sqrt{(1-\xi^2)}t_m+\varphi\right]$

and $x(t_{m+1})=Ce^{-\xi\omega_n t_{m+1}}\sin\left[\omega_n\sqrt{(1-\xi^2)}t_{m+1}+\varphi\right]$

$$\frac{x(t_m)}{x(t_{m+1})}=\frac{e^{-\xi\omega_n t_m}\sin\left[\omega_n\sqrt{(1-\xi^2)}t_m+\varphi\right]}{e^{-\xi\omega_n t_{m+1}}\sin\left[\omega_n\sqrt{(1-\xi^2)}t_{m+1}+\varphi\right]}$$

but since $\tan\left[\omega_n\sqrt{(1-\xi^2)}t+\varphi\right]=\dfrac{\sqrt{(1-\xi^2)}}{\xi}$ for $t=t_m$ and t_{m+1}

$$\sin\left[\omega_n\sqrt{(1-\xi^2)}t_{m+1}+\varphi\right]=\sin\left[\omega_n\sqrt{(1-\xi^2)}t_m+\varphi\right]$$

$$\frac{x(t_m)}{x(t_{m+1})}=\frac{e^{-\xi\omega_n t_{m+1}}}{e^{-\xi\omega_n t_m}}=e^{\xi\omega_n(t_{m+1}-t_m)}$$

and substituting for $t_{m+1}-t_m$ from (c),

$$\frac{x(t_{m+1})}{x(t_m)} = e^{\left(\frac{2\pi\xi}{\sqrt{(1-\xi^2)}}\right)}$$

The log decrement δ is defined as

$$\delta = \log_e\left(\frac{x(t_{m+1})}{x(t_m)}\right) = \left(\frac{2\pi\xi}{\sqrt{(1-\xi^2)}}\right)$$

If δ is measured we may determine ξ in terms of δ thus

$$\delta = \left(\frac{2\pi\xi}{\sqrt{(1-\xi^2)}}\right)$$

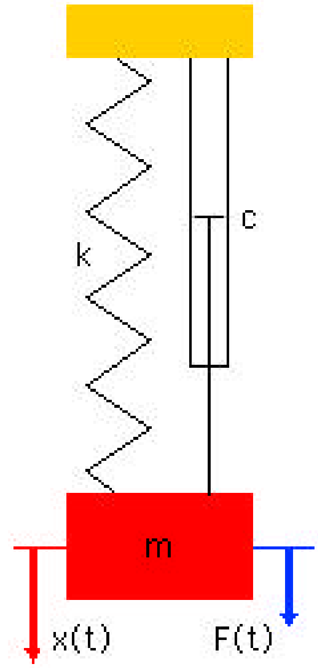
$$\delta^2 = \frac{4\pi^2\xi^2}{1-\xi^2}$$

$$\delta^2 - \xi^2\delta^2 = 4\pi^2\xi^2$$

$$\xi^2(4\pi^2 - \delta^2) = \delta^2$$

and hence $$\xi = \frac{\delta}{\sqrt{(4\pi^2 - \delta^2)}} \quad \text{............................} \quad (18)$$

<u>External Excitation</u>



If the mass in Figure 1 is subjected to a force F(t) acting in the positive x(t) then the equation of motion (2) becomes,

$$m\ddot{x}(t) + c\,\dot{x}(t) + kx(t) = F(t) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (19)$$

Taking Laplace Transforms

$$m[-\dot{x}(0) - sx(0) + s^2X(s)] + c[-x(0) + sX(s)] + kX(s) = \mathcal{L}F(t)$$

$$X(s) = \frac{\mathcal{L}F(t)}{ms^2 + cs + k} + \frac{m(\dot{x}(0) + sx(0)) + cx(0)}{ms^2 + cs + k} \quad \ldots\ldots\ldots\ldots\ldots \quad (20)$$

The latter term is identical to that already examined, i.e. there is a transient component of the response which is dependent on the initial conditions. The solution of x(t) is thus the superposition of a free vibration arising from the initial conditions plus a solution resulting from the applied force. We shall now examine this second component of the solution.

$$X(s) = \frac{\mathcal{L}F(t)}{ms^2 + cs + k}$$

Frequently F(t) is cyclic in nature. Thus the solution when F(t) = Fsinωt is commonly examined. This is also relevant since many excitation forces may be separated into sinusoidal components.

Now $$LF(t) = LF\sin\omega t = \frac{F}{s^2 + \omega^2}$$

therefore $$X(s) = \frac{F}{(s^2+\omega^2)(ms^2+cs+k)} \quad \text{................} \quad (21)$$

taking partial fractions

$$X(s) = \frac{As+B}{(s^2+\omega^2)} + \frac{Cs+D}{(ms^2+cs+k)} \quad \text{................} \quad (22)$$

where the second term will yield an exponential decaying motion of the kind already encountered. This is a transient motion arising from the fact that taking the Laplace Transform implies that the forcing function is zero when t < 0. Thus the sinusoidal excitation commences at t = 0 and there is a transient associated with the 'start up' of the motion. When considering sinusoidal excitation we are normally interested in the steady state motion that results and is given by the first term in equation (22). Thus we are interested in A and B. From equations (21) and (22) if the coefficients of s^3, s^2, s and the constant term are compared it is possible to show that,

$$A = \frac{-\omega cF}{(k-m\omega^2)^2+\omega^2c^2}$$

$$B = \frac{\omega(k-m\omega^2)F}{(k-m\omega^2)^2+\omega^2c^2}$$

The steady state component of x(t) is given by taking the inverse Laplace transform of,

$$X(s) = \frac{As+B}{(s^2+\omega^2)}$$

thus

$$X(t) = A\cos\omega t + \frac{B}{\omega}\sin\omega t$$

and substituting for A and B gives

$$x(t) = \frac{F\left(-\omega c\cos\omega t + (k-m\omega^2)\sin\omega t\right)}{(k-m\omega^2)^2+\omega^2c^2}$$

so that $$x(t) = \frac{F\sin(\omega t+\varphi)}{\left((k-m\omega^2)^2+\omega^2c^2\right)^{1/2}} \quad \text{................} \quad (23)$$

$$\text{where } \tan\varphi = \frac{-\omega c}{(k - m\omega^2)}$$

It is conventional to represent x(t) as Xsin(ωt+φ), where X is the amplitude of the response and φ the phase lag. Thus

$$X = \frac{F}{\left((k - m\omega^2)^2 + \omega^2 c^2\right)^{1/2}}$$

The results may be non-dimensionalised by multiplying throughout by k.

$$\text{Thus, } \frac{kX}{F} = \frac{1}{\left(\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\xi^2 \frac{\omega^2}{\omega_n^2}\right)^{1/2}} \quad \text{and } \tan\varphi = \frac{-2\xi\omega/\omega_n}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)} \quad \text{.... (24)}$$

The variation of kX/F and φ as functions of ω/ω_n are shown in Figure 6.

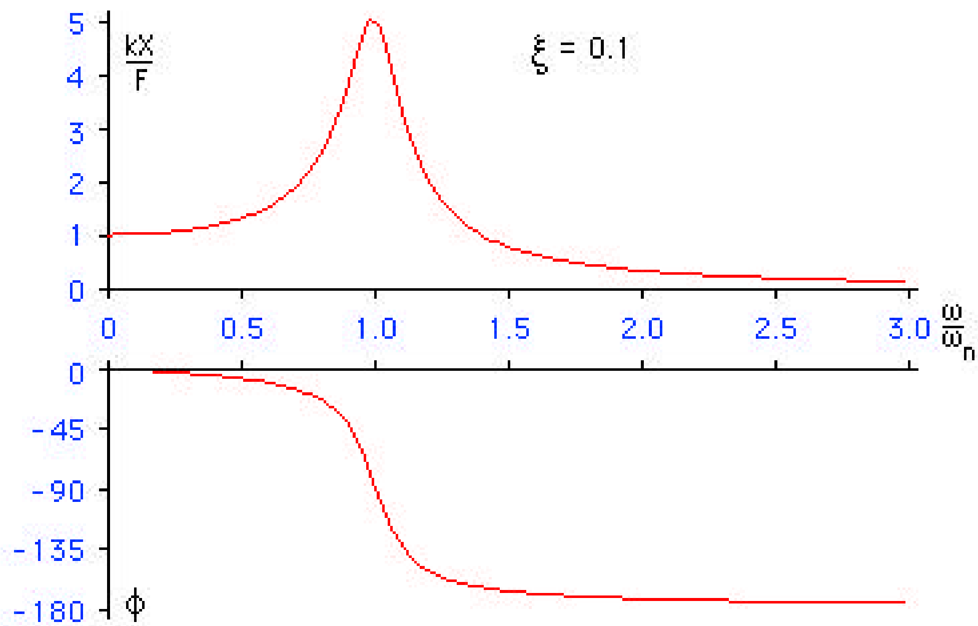


Figure 6 Steady state response; amplitude and phase.

The main points to note are that:-

i) $\frac{kX}{F} \to 1$ as $\frac{\omega}{\omega_n} \to 0$ this is generally known as the quasi-static condition as $\frac{X}{F} \to \frac{1}{k}$.

ii) $\frac{kX}{F} \to 0$ as $\frac{\omega}{\omega_n} \to \infty$.

iii) Resonance occurs, i.e. kX/F is a maximum, when d(kX/F)/dt is zero. This can be shown to be when ,

$$\omega_r = \omega_n \sqrt{(1-2\xi^2)}$$

However, note that there is no real solution for ω_r when $\xi > 1/\sqrt{2}$, i.e. the response continuously falls with frequency.

iv) The final point of interest is the response amplitude at resonance, i.e.

$$\frac{kX}{F} = \frac{1}{\left(\left(1-\left(1-2\xi^2\right)\right)^2 + 4\xi^2\left(1-2\xi^2\right)\right)^{1/2}}$$

$$\frac{kX}{F} = \frac{1}{\left(4\xi^4 + 4\xi^2 - 8\xi^4\right)^{1/2}}$$

which for small values of ξ is equal to $1/(2\xi)$.

The method of approach thus far has been to use Laplace Transforms. When the steady state response is required it is possible to obtain this more directly using an exciting force $Fe^{i\omega t}$, and assuming a steady state response $Xe^{i\omega t}$. This method does not provide the transient solutions due to initial conditions and the commencement of the excitation. Consider again the equation of motion (19), this becomes

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = Fe^{i\omega t} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (25)$$

substituting $x(t) = Xe^{i\omega t}$ gives

$$-m\omega^2 Xe^{i\omega t} + i\omega c Xe^{i\omega t} + kXe^{i\omega t} = Fe^{i\omega t}$$

and thus

$$\frac{X}{F} = \frac{1}{k - m\omega^2 + i\omega c}$$

This is a complex expression the amplitude of which is the response amplitude and the phase of which indicates the phase between displacement and force. If the equation is non-dimensionalised by multiplying throughout by k we obtain,

$$\frac{kX}{F} = \frac{1}{1 - \frac{\omega^2}{\omega_n^2} + i2\xi\frac{\omega}{\omega_n}}$$

and this has the same response amplitude and phase lag as before,

$$\frac{kX}{F} = \frac{1}{\left(\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\xi^2\frac{\omega^2}{\omega_n^2}\right)^{1/2}} \quad \text{and} \quad \tan\varphi = \frac{-2\xi\omega/\omega_n}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2}$$

It is clear that this method allows the steady state response to be obtained far more simply and thus it will be the preferred method for steady state analysis.

So far the excitation has been limited to an oscillating force of constant amplitude. There are, however, other types of excitation. Those normally considered are excitation by a rotating out-of-balance mass and also abutment (or floor) excitation.

<u>Out-of-balance excitation</u>

The excitation by an out-of-balance is shown diagrammatically in Figure 7. The out-of-balance mass m' is at a radius r and is rotating at an angular frequency ω, so that $\theta = \omega t$. This results in an excitation force being applied to the main mass m in the axial direction of

$$-m'\ddot{x}(t) + m'\omega^2 r\sin\omega t$$

The first term arises because m' is attached to the main mass and the second term because of the rotation (Full derivation).

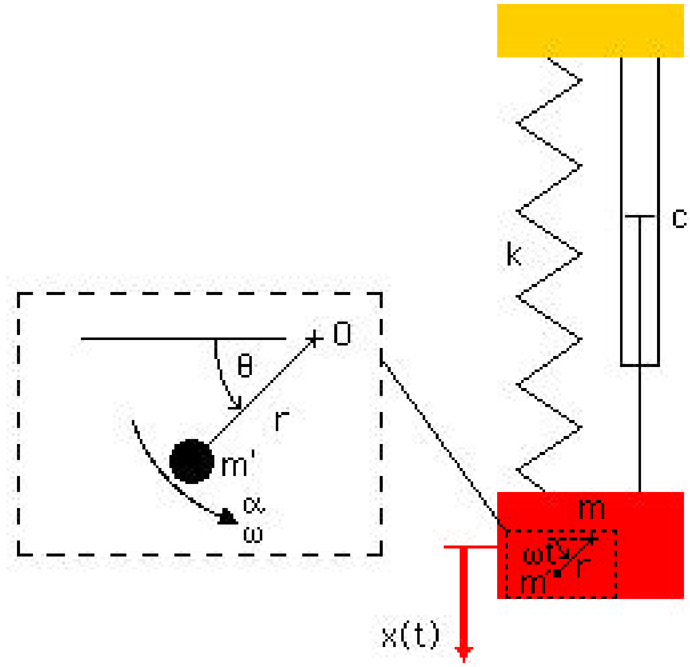


Figure 7 Excitation by an out-of-balance mass.

The equation of motion thus becomes,

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = -m'\ddot{x}(t) + m'\omega^2 r\sin\omega t \quad .... \quad (26)$$

which may be rearranged to give

$$(m + m')\ddot{x}(t) + c\dot{x}(t) + kx(t) = m'\omega^2 r\sin\omega t$$

This is basically the same as for the excitation by $F\sin\omega t$, but F is replaced by $m'\omega^2 r$ and the mass of the system is increased to m + m'. The steady state response is thus given by comparison with equation (23), so that

$$x(t) = \frac{m\omega^2 r\sin(\omega t - \varphi)}{\left([k - (m + m')\omega^2]^2 + \omega^2 c^2\right)^{1/2}} \quad \text{where } \tan\varphi = \frac{-\omega c}{[k - (m + m')\omega^2]}$$

The results may also be non-dimensionalised thus,

$$X = \frac{\dfrac{m'r}{m}\dfrac{\omega^2}{\omega_n^2}}{\left(\left(1 - \dfrac{\omega^2}{\omega_n^2}\right)^2 + 4\xi^2\dfrac{\omega^2}{\omega_n^2}\right)^{1/2}} \quad \text{and } \tan\varphi = \frac{-2\xi\omega/\omega_n}{\left(1 - \dfrac{\omega^2}{\omega_n^2}\right)}$$

where

$\omega_n = \sqrt{k/(m + m')}$

and

$\xi = c/2\sqrt{k(m + m')}$

Typical response curves are shown in Figure 8. The main points to note are that:-

i) $X \rightarrow 0$ as $\dfrac{\omega}{\omega_n} \rightarrow 0$ this is because the excitation force tends to zero.

ii) $X \rightarrow \dfrac{-m' r}{m + m'}$ as $\dfrac{\omega}{\omega_n} \rightarrow \infty$ This is antiphase with the excitation and the centre of mass of m and m' does not move.

iii) Resonance occurs, i.e. X is a maximum, when d(X)/dt is zero. This can be shown to be when $\omega_r = \omega_n/\sqrt{(1-2\xi^2)}$. However, note that there is no real solution for ω_r when $\xi>1/\sqrt{2}$, that is the response continuously rises with frequency and approaches m'r/(m + m') asymptotically. When there is a resonance it should be noted that ω_r is greater than ω_n.

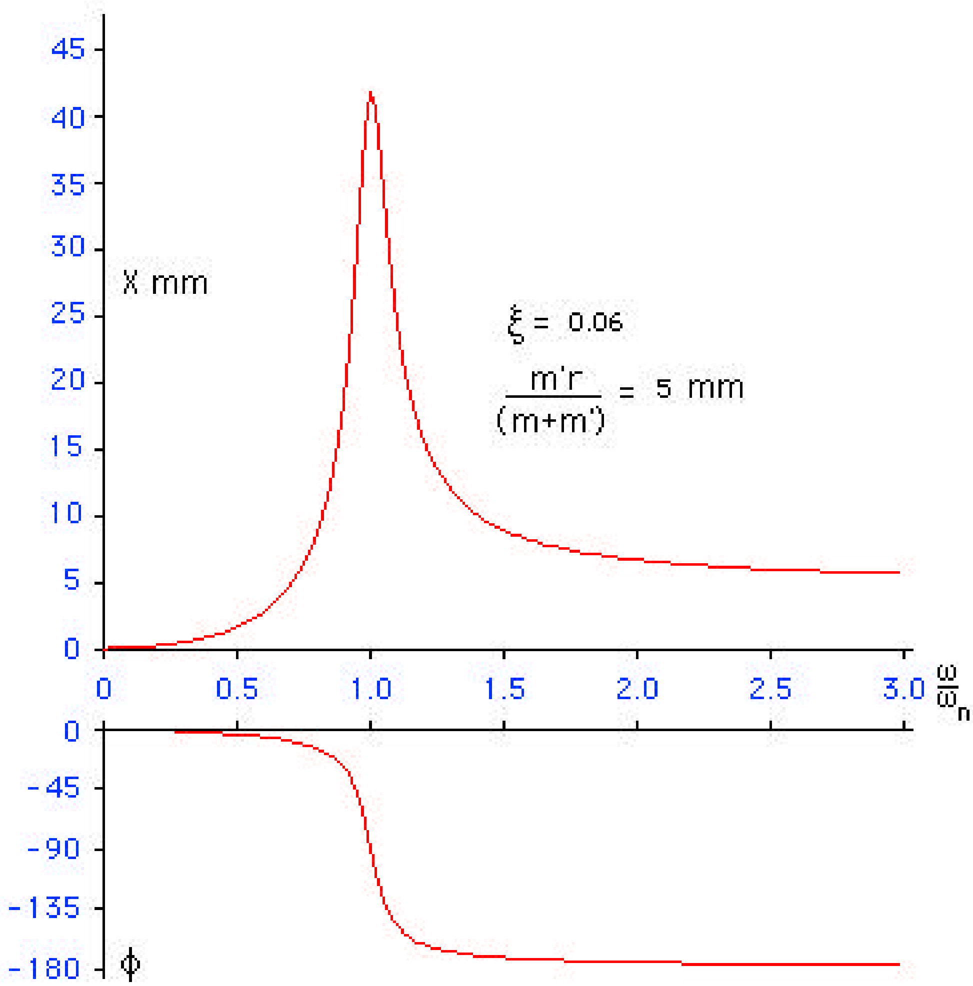


Figure 8 Steady state response to out-of -balance; amplitude and phase.

Finally it should be remembered that there would be transients arising from getting up to speed and also from the initial displacement and velocity.

<u>Abutment Excitation</u>

Sinusoidal excitation by the abutment is shown diagrammatically in Figure 9.

Applying Newton's second law of motion gives,

$$m\ddot{x}(t) = k\left(x_0 - x(t)\right) + c(\dot{x}_0 - \dot{x}(t)) \quad \ldots\ldots \quad (27)$$

For this example it is much simpler to find the steady state response by taking the input excitation to be $x_0 = X_0 e^{i\omega t}$ and the response to be $x(t) = Xe^{i\omega t}$. On substituting in (27) this gives,

$$(k - m\omega^2 + i\omega c)Xe^{i\omega t} = (k + i\omega c)X_0 e^{i\omega t}$$

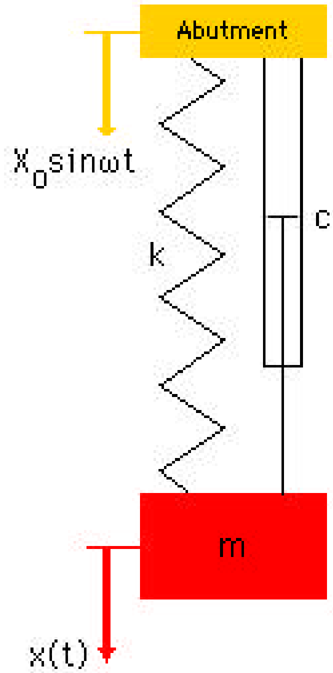


Figure 9 Abutment excitation

rearranging,

$$\frac{X}{X_0} = \frac{(k + i\omega c)}{(k - m\omega^2 + i\omega c)}$$

The phase relationship is very complex but normally we would be interested in the amplitude of the response and this may be found by finding the amplitude of the 'top' and 'bottom' vectors. If the result is also non-dimensionalised by dividing throughout by k, the following result is obtained,

$$\frac{X}{X_0} = \frac{\left(1 + 4\xi^2 \dfrac{\omega^2}{\omega_n^2}\right)^{1/2}}{\left(\left(1 - \dfrac{\omega^2}{\omega_n^2}\right)^2 + 4\xi^2 \dfrac{\omega^2}{\omega_n^2}\right)^{1/2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (28)$$

Typical response curves are shown in Figure 10.

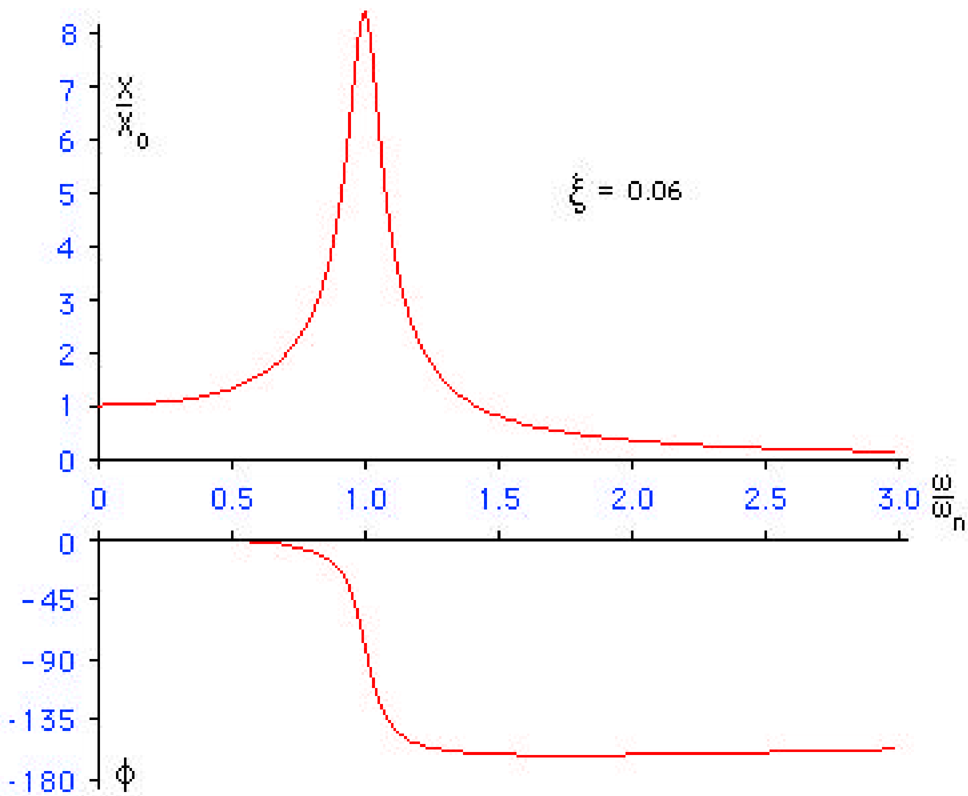


Figure 10 Steady state response to abutment excitation; amplitude and phase.

The main points to note are that:-

i) $\frac{X}{X_0} \to 1$ as $\frac{\omega}{\omega_n} \to 0$, that is there is no deflection across the spring.

ii) $\frac{X}{X_0} \to 1$ as $\frac{\omega}{\omega_n} \to \frac{1}{\sqrt{2}}$ independent of the value of ξ.

iii) $\frac{X}{X_0} \to 0$ as $\frac{\omega}{\omega_n} \to \infty$, this is termed vibration isolation.

iv) Resonance occurs, i.e. X/X_0 is a maximum, when d(X)/dt is zero. This can be shown to be when
$\omega_r^2 = \omega_n^2(-1+\sqrt{(1+8\xi^2})/4\xi^2$. Note that this resonance occurs for all values of ξ.

Transmissibility

When the system shown in Figure 1 is excited by an oscillating force it is often of interest to determine the force on the abutment that results from the motion. This is important since it is this force that may induce floor borne vibrations and cause problems elsewhere. Transmissibility is defined as the amplitude of the force on the abutment divided by the exciting force amplitude.

The force on the abutment is given by,

$$F_T e^{i\omega t} = kx(T) + c\dot{x}(t)$$

when the excitation force is $Fe^{i\omega t}$

then $x(t) = Xe^{i\omega t}$ and hence $F_T = (k + i\omega c)X$. Substituting for X from equation (24) gives,

$$\frac{F_T}{F} = \frac{\left(1 + 4\xi^2 \frac{\omega^2}{\omega_n^2}\right)^{1/2}}{\left(\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\xi^2 \frac{\omega^2}{\omega_n^2}\right)^{1/2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (29)$$

This is the same result as for abutment motion given in equation (28). Thus Figure 10 shows the variation of transmissibility with frequency as well as X/X_0.

Measures of Damping

One of the areas of confusion in the vibration area is the multiplicity of methods of expressing damping. So far we have encountered:-

i) c the viscous damping coefficient.
ii) ξ the viscous damping ratio defined as $\xi = c/2\sqrt{km}$
iii) δ the logarithmic decrement, defined as $\delta = 2\pi/\sqrt{(1- \xi^2)}$

There are also several others which we will need to consider.

Dynamic magnification factor (Q)

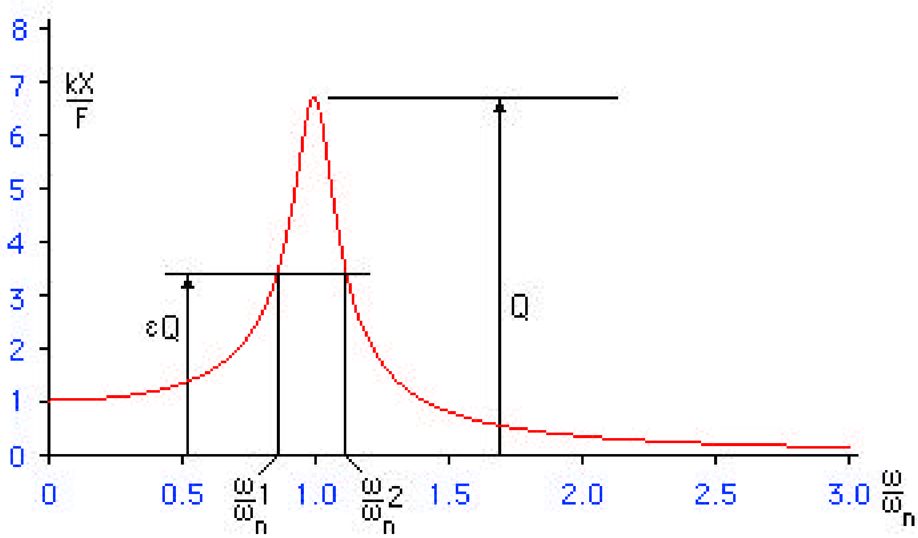


Figure 11 Typical response curve.

The ratio of the amplitude at resonance to the quasi-static amplitude when excited by an oscillating force is termed the dynamic magnification factor Q. For lightly damped structures Q = 1/(2ξ). This is a particularly useful measure as it may be determined from the steady state response curve using the bandwidth of the resonance peak. Figure 11 shows a typical resonance curve and at the frequencies ω_1 and ω_2 the response is ε times the response at resonance. Now ω_2 - ω_1 is called the bandwidth (Δω) and may be used to determine Q. As has been shown previously,

$$\frac{kX}{F} = \frac{1}{\left(\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\xi^2 \frac{\omega^2}{\omega_n^2}\right)^{1/2}}$$

and at resonance this is 1/(2ξ). Thus to find ω_1 and ω_2 when the response is ε times that at resonance we need the condition that,

$$\frac{\varepsilon}{2\xi} = \frac{1}{\left(\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\xi^2 \frac{\omega^2}{\omega_n^2}\right)^{1/2}}$$

rearranging gives

$$\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\xi^2 \frac{\omega^2}{\omega_n^2} = \frac{4\xi^2}{\varepsilon^2}$$

if we put $q = \omega/\omega_n$ then

$$q^4 + q^2(4\xi^2 - 2) + 1 - \frac{4\xi^2}{\varepsilon^2} = 0$$

$$q^2 = \frac{-(4\xi^2 - 2) \pm \sqrt{(4\xi^2 - 2)^2 - 4\left(1 - \frac{4\xi^2}{\varepsilon^2}\right)}}{2}$$

$$q^2 = 1 - 2\xi^2 \pm 2\xi\sqrt{\xi^2 - 1 + \frac{1}{\varepsilon^2}}$$

Now if ξ is small and ε is not too near unity,

$$q^2 = 1 \pm 2\xi\sqrt{\frac{1}{\varepsilon^2} - 1}$$

thus $$\frac{\omega_1^2}{\omega_n^2} = 1 - 2\xi\sqrt{\frac{1}{\varepsilon^2} - 1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (30)$$

and $$\frac{\omega_2^2}{\omega_n^2} = 1 + 2\xi\sqrt{\frac{1}{\varepsilon^2} - 1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (31)$$

Now $$\frac{\omega_2 - \omega_1}{\omega_n} = \frac{\omega_2^2 - \omega_1^2}{\omega_n(\omega_2 + \omega_1)}$$

and $\omega_2+\omega_1$ is approximately equal to $2\omega_n$ therefore (31) - (32) gives,

$$\frac{\Delta\omega}{\omega_n} = \frac{\omega_2 - \omega_1}{\omega_n} = \frac{\omega_2^2 - \omega_1^2}{2\omega_n^2} = \frac{\left(1 + 2\xi\sqrt{\frac{1}{\varepsilon^2} - 1}\right) - \left(1 - 2\xi\sqrt{\frac{1}{\varepsilon^2} - 1}\right)}{2} = 2\xi\sqrt{\frac{1}{\varepsilon^2} - 1}$$

It is common to measure $\Delta\omega$ when $\varepsilon = 1/\sqrt{2}$ (this is called the 3dB point as 20log(1/√2) = -3) and also to approximate $\omega_r = \omega_n$ therefore

$$\frac{\Delta\omega}{\omega_r} = 2\xi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (32)$$

This is a very useful measure of damping as it may be applied to a response with several resonances and will give the equivalent damping ratio for each resonance.

<u>Specific damping capacity(ϕ) and loss factor (η)</u>

Another common basis for measuring damping involves energy loss and involves the determination of the energy lost per cycle. In all cases involving damping the force - displacement relationship when plotted graphically will enclose an area, commonly called the hysteresis loop. The work done during a cycle is given by,

$$W_d = \int F_d dx$$

which for viscous damping gives (since the other force components do no net work during a cycle)

$$W_d = \int c\dot{x}\,dx = \int c\frac{dx}{dt}dx = \int c\frac{dx}{dt}\frac{dx}{dt}dt = \int c\dot{x}^2\,dt$$

Now for cyclic motion $x = X\sin(\omega t - \varphi)$ and therefore $\dot{x} = \omega X\cos(\omega t - \varphi)$ and thus for one cycle,

$$W_d = \int_0^{2\pi/\omega} c(\omega X\cos(\omega t - \varphi))^2 dt = c\omega^2X^2\int_0^{2\pi/\omega}(\cos(\omega t - \varphi))^2 dt$$

$$= c\omega^2X^2\int_0^{2\pi/\omega}\left(\frac{1+\cos 2(\omega t - \varphi)}{2}\right)dt$$

$$= c\omega^2X^2\left|\frac{t}{2} + \frac{\sin 2(\omega t - \varphi)}{4\omega}\right|_0^{2\pi/\omega}$$

therefore $W_d = \pi c\omega X^2$

The main interest involves the energy lost per cycle at resonance, which for small levels of damping requires, $\omega = \sqrt{k/m}$ and $c = 2\xi\sqrt{km}$. This gives

$$W_d = 2\pi\xi kX^2 \quad \text{..........................} \quad (33)$$

Now the specific damping capacity ϕ is defined as the energy loss per cycle divided by the peak energy stored, which for the spring mass system being considered is the energy stored in the spring at maximum deflection, i.e. $kX^2/2$.

$$\text{Therefore } \phi = \frac{4\pi\xi kX^2}{kX^2} = 4\pi\xi \quad \text{..................} \quad (34)$$

The main drawback to the above is that the damping force is dependent on frequency and thus a specific frequency is required in order to evaluate the energy loss. In practice the energy loss per cycle is often not too dependent on frequency and a hysteretic damping factor is introduced, which is defined as $h = \omega c$ so that at any frequency

$$W_d = \pi h X^2$$

In this case, the specific damping capacity ϕ is then

$$\phi = \frac{2\pi h X^2}{k X^2} = 2\pi \frac{h}{k} = 2\pi\eta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (35)$$

where η=h/k and is defined as the loss factor.

If we combine all the definitions of damping the following relationships are obtained,

$$\phi = 2\pi\eta = 2\pi\xi = 2\pi / Q = 2\delta = 2\pi\Delta\omega / \omega_r$$

where $\Delta\omega$ is measured at the 3dB point